Artificial Intelligence

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Solving Problems by Searching

Suggested textbook

S. Russell, P. Norvig, *Artificial Intelligence – A Modern Approach*, 4th Ed., Pearson, 2021 (or a previous edition)

Introduction

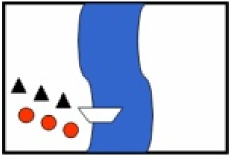
Some example problems are presented in the following.

Assume that your goal is to design a **rational agent**, in the form of a computer program, capable of **autonomously** solving them.

Remember: a rational agent is a system that **acts rationally**, according to a **well-defined objective**.

Missionaries and cannibals

A classic AI toy-problem: three missionaries and three cannibals must cross a river on a boat that can only hold two people, without leaving more cannibals than missionaries on either side of the river. How can all six get across the river safely?



Game playing: 15-puzzle

Another classic AI's toy-problem: transform an array of tiles from an initial configuration into a given, desired configuration, by a sequence of moves of a tile into an adjacent empty cell.

A more challenging goal: find the **shortest** of such sequences.

An example:

13	10	11	6
5	7	4	8
1		14	9
3	15	2	12

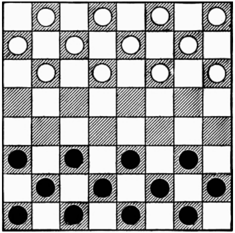
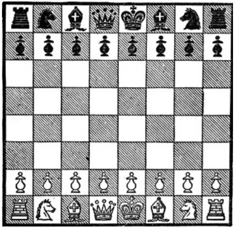
initial configuration

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

desired configuration

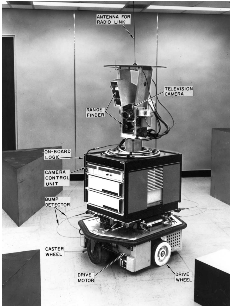
Game playing: checkers and chess

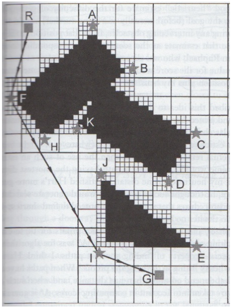
Two historical problems addressed by many researchers since the early days of AI (chess has been named the "Drosophila of AI").

Robot navigation

A real-world problem addressed since mid–1960s.



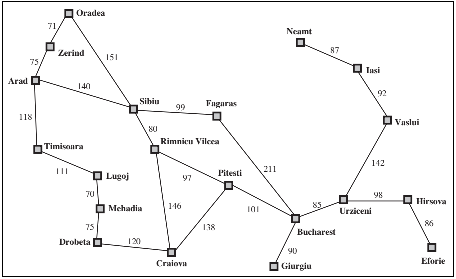


Left: Shakey the robot (1968). Right: formalisation of a navigation problem for Shakey: finding a route from R to G made up of straight line segments, possibly the **shortest** one, avoiding obstacles (black polygons).

Route finding in maps

Find a route from Arad to Bucharest using the information shown in the map below.

A more challenging version: find the **shortest** route.



Common features of the above problems

Although the above problems may seem very different from each other, they share some high-level features that allow one to solve them using the **same** approach.

Main feature: a clear **goal** can be defined, as a set of desired "world states", together with a set of available "actions", each one leading from one state to another.

Once the goal is defined, the task is to **search** for a **sequence** of **actions** that lead to a **goal state**. Hence the name "search problem".

This requires one to suitably define the **actions** and the **states** of a given problem.

A framework for search problems

Problems exhibiting the above characteristics can be formalized as follows:

1. **Goal formulation**: what are the desired "world states"?
2. **Problem formulation**:
   - what are the **actions** to consider?
   - what are the **states** to consider?

   The crucial point in this step is to find a proper level of **abstraction**, by removing irrelevant details.

Under the above formulation:

- the **solution** of a problem is a **sequence of actions** that lead to a goal state
- the process of looking for a solution is called **search**

Goal and problem formulation: examples

15-puzzle

13	10	11	6
5	7	4	8
1		14	9
3	15	2	12

initial configuration

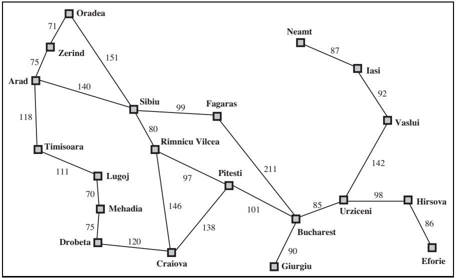
1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

desired configuration

- **goal**: getting to the desired tile configuration (possibly, by the shortest sequence of moves)
- **states**: each possible 16! tile configurations
- **actions**: moving the n-th tile ($n = 1, \ldots, 15$) to one of the adjacent cells (two, thee or four), if empty

Goal and problem formulation: examples

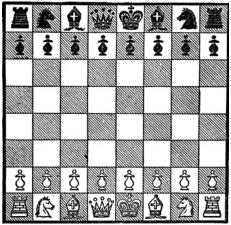
Route finding in maps



- **goal**: getting from a given city to a destination one (possibly, through the shortest route)
- **states**: being in each possible city
- **actions**: moving between two adjacent cities

Goal and problem formulation: examples

Chess



- **goal**: to checkmate (this goal is achieved in **many** possible chessboard configurations, i.e., there are many goal states)
- **states**: each legal chessboard configuration
- **actions**: all legal moves

Properties of search problems

- **Static** vs **dynamic**: does the environment change over time? Examples: 15-puzzle and chess are static; robot navigation is dynamic, if the position of obstacles changes over time
- Fully vs partially **observable**: is the current state completely known? Examples: 15-puzzle and chess are fully observable; robot navigation is partially observable, if sensors are not "perfect"
- **Discrete** vs **continuous** sets of states and actions. Examples: 15-puzzle and chess are discrete, robot navigation is continuous
- **Deterministic** vs **non-deterministic**: is the outcome (the resulting state) of any sequence of actions certain. i.e., known in advance? Examples: 15-puzzle is deterministic, chess is not (due to the opponent's move, that is unknown when deciding one's own)

Examples of real-world search problems

Many challenging real-world problems can be formulated as search problems. Some examples:

- **traveling salesperson problem**: finding the **shortest** tour that allows one to visit every city of a given map **exactly once** (applications to planning, logistics, microchip manufacture, DNA sequencing, etc.)
- **route-finding**: routing in computer networks, airline travel planning, etc.
- **VLSI design**: cell layout, channel routing

Search problems: formal defintion

How to devise algorithms, and how to implement them using some programming language, to solve search problems?

First, a rigorous **problem definition** is needed.

The goal and problem formulation sketched above can be formally defined in terms of four components:

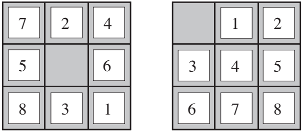
- the **initial state**
- the set of possible **actions**
- the **goal test**
- the **path cost**

Search problems: formal defintion

- **Initial state**: a description of the state where the search starts
- **Actions**: a description of all possible actions available at any possible state. They can be defined as a **successor function** SF which, given a state s, returns the set of ordered pairs (a', s'), where a' is a legal action in state s and s' the resulting state
  - the initial state and SF **implicitly** define the **state space**: a **graph** whose nodes correspond to states and edges to actions
  - a **path** in the state space is a sequence of states connected by a sequence of actions
- **Goal test**: a function that determines whether or not any given state is a goal state
- **Path cost** function: it assigns a numeric cost to each path. In many problems it equals the sum of the costs of the individual actions (**step cost**) along the path

Example: 8-puzzle

A simpler version of the 15-puzzle problem.



Start State Goal State

- **States**: all possible board configurations (i.e., the location of each tile in the board)
- **Initial state**: any given board configuration (note that only **half** of the configurations can be reached by any given one)
- **Goal test**: checking whether the input state matches the desired one
- **Path cost**: if the goal is to reach the desired configuration by the **shortest** sequence of moves, each action “costs” 1 move, and the path cost is the number of steps (moves) in the path

Example: 8-puzzle

- **Actions**: different descriptions are possible, e.g.:
  - moving **the *n*-th tile** ($n \in \{1, \ldots, 8\}$) to one of the empty adjacent cells, if any (e.g., "move tile 3 right")
  - moving **the blank** to one of the adjacent cells (e.g., "move the blank down")

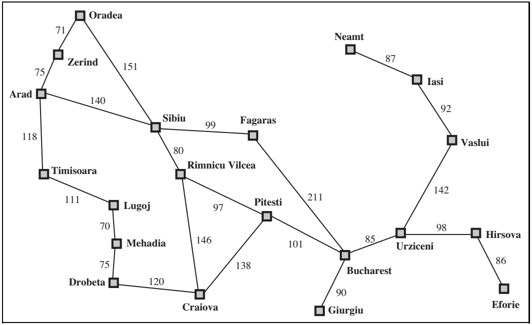
  Using the latter description, the input/output behaviour of SF can be represented as in the following example:

  $$SF\left(\begin{array}{|c|c|c|}\hline 7 & 2 & 4 \\ \hline 5 & & 6 \\ \hline 8 & 3 & 1 \\ \hline\end{array}\right) = \left\{\left(\text{“move the blank down”}, \begin{array}{|c|c|c|}\hline 7 & 2 & 4 \\ \hline 5 & 3 & 6 \\ \hline 8 & & 1 \\ \hline\end{array}\right), \ \ldots\right\}$$

The resulting state space is a graph made up of $\frac{9!}{2}$ nodes (possible board configurations reachable from any initial state).

Note that in this problem it is not convenient nor necessary to compute and store the state space beforehand into computer memory: it is **implicitly** defined by the initial state and SF.

Example: route finding in maps



Example: route finding in maps

- **States**: the set of cities in the map
- **Initial state**: any given city in the map
- **Goal test**: checking whether an input city is the destination one
- **Actions**: moving from any given city to an adjacent one. Accordingly, *SF* returns all the cities adjacent to a given one (no further action description is necessary for this problem), e.g.:
  $SF(\mathrm{Arad}) = \{(\mathrm{Timisoara},\ \mathrm{Sibiu},\ \mathrm{Zerind})\}$
- **Path cost**: if the goal is to find the **shortest** route to the destination, the cost of each action is the length (e.g., in km) of the route between the corresponding cities, and the path cost is the sum of the lengths of all routes in the path

Note that in this kind of problem the state space corresponds to the map. Accordingly, storing in computer memory the information on the map amounts to **explicitly** storing the whole state space.

Solving a search problem

From now on we shall consider the simplest kind of search problem: **static**, **fully observable**, **discrete**, **deterministic** (e.g., 8/15-puzzle and route finding in maps).

Key feature of this kind of problem: the search for a solution can be made **offline**, i.e., before starting the execution of the corresponding actions.

The main steps for solving such problems are therefore:

1. goal and problem **formulation** (discussed above)
2. **searching** for a solution (**offline**)
3. **executing** the actions

In the following we shall focus on the **search** step.

Search algorithms

A possible approach for solving a search problem: systematically "exploring" **all** the possible sequences of actions (i.e., all the paths in the state space) from the initial state, in a suitable order, until a goal state is reached.

This can be achieved by **iteratively** constructing the corresponding action sequences, by **expanding** at each iteration **one** of the current sequences, i.e., adding to it **every** possible **single** action.

During this process, the current set of action sequences can be conveniently represented by a **tree graph**, named **search tree**.

The criterion used for choosing **which** action sequence to expand at teach iteration defines what is called a **search strategy**.

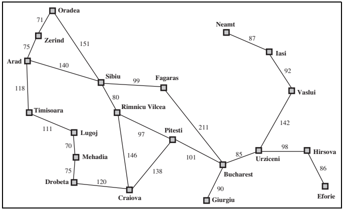
Search tree

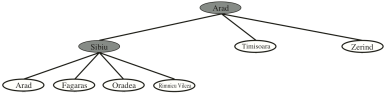
A **search tree** represents a set of action sequences (**partial solutions**) starting from the initial state:

- each **node** represents a state
- an **edge** represents the action that leads from the parent node's state to the child node's state
- a **leaf node** corresponds to the end state of an action sequence
- each sequence of nodes from the root to a leaf is called **path**, and represents one of the action sequences encoded in the search tree
- the **depth** of a node is the number of actions in the path from the root to that node (the root node has **zero** depth)
- the set of **leaf nodes** is called **fringe** or **frontier**

Note that a state can appear in **different** nodes of a search tree.

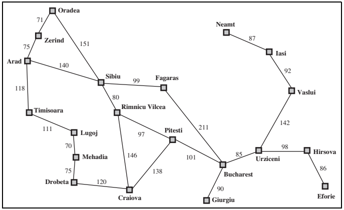
Search tree: an example

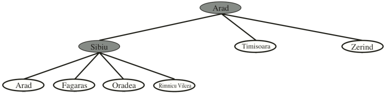




- Root node: the starting city, Arad
- Six leaf nodes (fringe, in white): six **partial** solutions
  - Arad $\rightarrow$ Sibiu $\rightarrow$ Arad
  - Arad $\rightarrow$ Sibiu $\rightarrow$ Fagaras
  - ...
- Edges: moving from a city to an adjacent one

State space and search tree





Note that a search tree is **different** from the state space.

One key difference is that every node corresponds to a **single** state of the state space, but every state may appear in several nodes of the search tree (i.e., in different paths).
In the example above, this is the case of the state 'Arad'.

Sketch of a general tree-search algorithm

1. construct the root node R of the search tree, associate the initial state to R, and set the fringe equal to {R} (the initial state is the only partial solution at this point)

2. repeat:

   2.1 if the fringe is empty, then no solution has been found and the algorithm stops

   2.2 choose **one** of the partial solutions, i.e., one leaf node N from the fringe (in the first iteration only R can be selected)

   2.3 if N contains a goal state, then the search is successfully completed: the algorithm stops, returning the sequence of actions in the path from R to N as the solution

   2.4 **expand** the state in N:

      2.4.1 apply *SF* to the state in N

      2.4.2 for each state **generated** by *SF*, construct a new leaf node, add it to the tree as a child of N, and add it to the fringe

      2.4.3 remove N from the fringe

Sketch of a general tree-search algorithm

Note that the above tree-search algorithm is **independent** of the search problem.

The key point is the choice of a leaf node to **expand** in step 2.2:

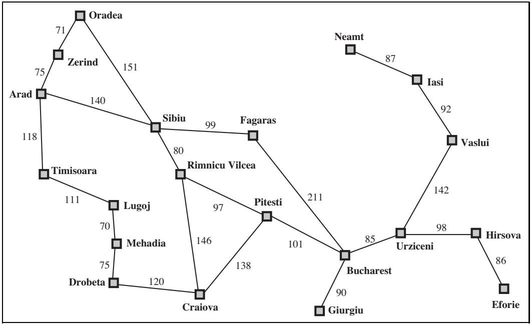
- **different** criteria can be used for this choice
- each criterion defines a specific **search strategy**
- each search strategy leads to a different **search algorithm**

It turns out that different search strategies can have very different **performance**, in terms of:

- **effectiveness**: the capability of finding a “good” solution
- **efficiency**: the amont of computing resources required to find a solution (processing time and memory requirements)

Tree-search algorithm: an example

Getting from Arad to Bucharest (route finding on maps)



Tree-search algorithm: an example

Step 1: the root node is constructed, coresponding to the initial state (Arad).

(a) The initial state

Tree-search algorithm: an example

Step 2: the first iteration starts.

Step 2.1: the fringe is not empty

Step 2.2: the fringe contains a single leaf node, the root node (Arad), which is therefore selected

Step 2.3 (goal test): Arad is not the desired state

Step 2.4: the chosen leaf (the root node) has to be **expanded**

(a) The initial state

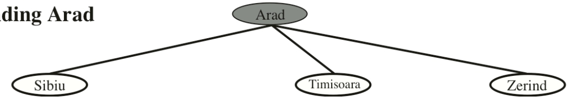
Tree-search algorithm: an example

Step 2.4.1: the successor function is applied to the state Arad, which **generates** all the states reachable from Arad

Step 2.4.2: the newly generated states are added as child nodes to the root node, and to the fringe

Step 2.4.3: the expanded node is removed from the fringe

(b) After expanding Arad



Leaf nodes, not yet expanded (the fringe), are shown in white; non-leaf nodes, already expanded (no more in the fringe) are shaded.

Tree-search algorithm: an example

Step 2: a new iteration starts.

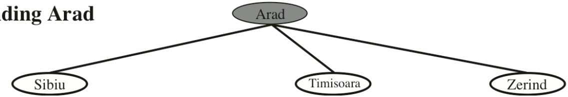
Step 2.1: the fringe is not empty

Step 2.2: the fringe contains three leaf nodes: **which one** to choose? This is decided by a **search strategy** (see later). For now, assume that Sibiu is chosen.

Step 2.3 (goal test): Sibiu is not the desired state

Step 2.4: the chosen leaf (Sibiu) has to be **expanded**

(b) After expanding Arad



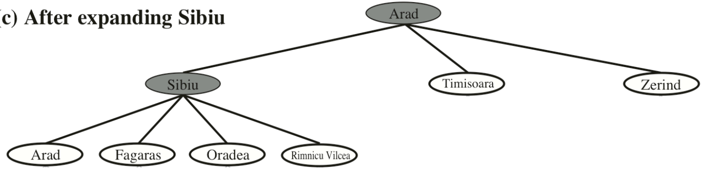
Tree-search algorithm: an example

Step 2.4.1: the successor function is applied to the state Sibiu, which **generates** all the states reachable from Sibiu (**including** Arad)

Step 2.4.2: the newly generated states are added as child nodes to the expanded node, and to the fringe

Step 2.4.3: the expanded node is removed from the fringe

(c) After expanding Sibiu



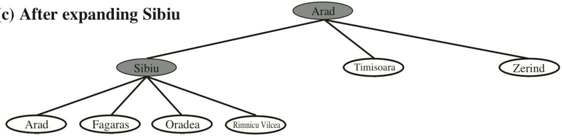
Then a new iteration starts . . .

Tree-search algorithm: an example

Note that the current search tree contains **six** action sequences (or partial solutions), corresponding to the paths from the root to each of the **six** leaf nodes:

1. Arad $\rightarrow$ Sibiu $\rightarrow$ Arad
2. Arad $\rightarrow$ Sibiu $\rightarrow$ Fagaras
3. Arad $\rightarrow$ Sibiu $\rightarrow$ Oradea
4. Arad $\rightarrow$ Sibiu $\rightarrow$ Rimnicu Vilcea
5. Arad $\rightarrow$ Timisoara
6. Arad $\rightarrow$ Zerind

(c) After expanding Sibiu



General tree-search algorithm

A more concise (but still informal) description:

function TREE-SEARCH (`problem`, `strategy`)
returns a solution, or failure
    construct the root node using the initial state of `problem`
    loop do
        if there are no leaf nodes
            then return failure
        choose a leaf node according to `strategy`
        if the chosen leaf node contains a goal state
            then return the corresponding solution
        expand the chosen leaf node

Implementation hints

In the following, a more formal version of the above tree-search algorithm is presented as a pseudo-code, together with the corresponding data structures.

Note that this version is **independent** of the search problem, and of the programming language.

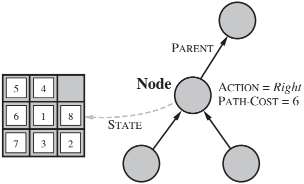
Details may change depending on the specific programming language used for implementing the algorithm.

Data structure: nodes of the search tree

The example below depicts the information that needs to be stored in computer memory to represent a **node** of the search tree:

- the **state** associated to the node
- the **parent** node (a reference, or pointer), useful to reconstruct the **path** from the root when a goal state is found
- the **action** that lead to this node from the parent one
- the **path cost** from the root node to this one

Additional information may be useful, e.g., the node **depth**.



Data structures: nodes of the search tree

The information outlined above can be conveniently stored into a **record** data structure, like the C language's `struct`, a dictionary in Python, or into instance variables of a **class** in object–oriented languages, including the following fields:

STATE	a problem-dependent representation of the corresponding state
PARENT-NODE	a reference or pointer to the parent node
ACTION	a description of the action that lead from the parent node to this one
PATH-COST	the total cost of the actions on the path from the root to this node
DEPTH	the number of actions in the path from the root to this node

Data structures: fringe of the search tree

At each step of the tree-search algorithm one of the leaf nodes (i.e., an element of the fringe) must be selected for expansion. The choice among all the leaf nodes is made according to a given **search strategy**, as discussed later.

The leaf nodes must therefore be **quickly** accessible. To this aim, references or pointers to leaf nodes should also be stored into a **linear** data structure, like a **linked list** in C or a **list** or **set** in Python.

A convenient solution is to store pointers to each node in the fringe in a **queue**, a **first-in first-out** (FIFO) data structure.

Newly generated nodes are therefore inserted into the queue in the order in which they will be expanded by the chosen search strategy. This way, the next node to be expanded is always the **first** node in the current fringe.

Data structures: the search problem

Information specific to the search problem at hand can be stored, e.g., into a record data structure:

INITIAL-STATE	a problem-dependent representation of the initial state
GOAL-TEST	a function that checks whether a given state is a goal state
SUCCESSOR-FN	the function *SF* (see above) that returns a set of pairs $(action, state)$ from a given state
STEP-COST	a function that returns the cost of carrying out a given action from a given state

For instance, in C language the values of the GOAL-TEST, SUCCESSOR-FN and STEP-COST fields can be **pointers to functions**, whereas in Python they can be **references** to functions (i.e., function names).

Implementation of the tree-search algorithm

A **possible** implemementation of the tree-search algorithm is shown in the next slide.

Note that this implementation is **problem-independent**, and is an example of **modular** programming style:

- it can be used for **any** search problem
- all the **problem-specific** details (e.g., the data structure representing a state and the goal-test function) are represented or implemented **separately**

Function and field names are written in SMALL CAPITALS.
The notation FIELD-NAME[`record`] denotes the value of the field FIELD-NAME of a `record`.

Implementation of the tree-search algorithm

```
function TREE-SEARCH (problem, ENQUEUE)
returns a solution, or failure
  fringe ← an empty queue
  fringe ← ENQUEUE(MAKE-NODE(INITIAL-STATE[problem]),
                            fringe)
  loop do
    if EMPTY?(fringe) then return failure
    node ← REMOVE-FIRST(fringe)
    if GOAL-TEST[problem](STATE[node]) succeeds
    then return SOLUTION(node)
    fringe ← ENQUEUE(EXPAND(node, problem), fringe)
```

The search strategy is assumed to be defined through the function ENQUEUE, which is passed to TREE-SEARCH as an argument (e.g., a pointer to a function in C language). As explained above, ENQUEUE inserts the newly generated nodes in a queue, in the order in which they have to be expanded according to the corresponding search strategy.

Implementing auxiliary functions: node expansion

This function expands a node, connects its children (leaf) nodes to it, and returns them:

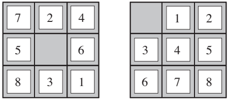
function EXPAND(`node`, `problem`) **returns** a set of nodes
    `successors` $\leftarrow$ the empty set
    for each $\langle$`action`, `result`$\rangle$ **in**
            SUCCESSOR-FN[`problem`](STATE[`node`]) **do**
        `n` $\leftarrow$ MAKE-NODE(`result`)
        PARENT-NODE[`n`] $\leftarrow$ `node`
        ACTION[`n`] $\leftarrow$ `action`
        PATH-COST[`n`] $\leftarrow$ PATH-COST[`node`] +
                        STEP-COST[`problem`](`node`, `action`)
        DEPTH[`n`] $\leftarrow$ DEPTH[`node`] + 1
        add `n` to `successors`
    return `successors`

Other auxiliary functions

- MAKE-NODE(`s`) returns a new instance of the node data structure, storing the state `s` in its STATE field, with no parent, no action, zero depth and zero path-cost
- REMOVE-FIRST(`q`) removes the first element from the queue `q` and returns it
- SOLUTION(`n`) returns the sequence of actions (the values of the ACTION fields) from the root of the tree to node `n`, following the pointers in the PARENT-NODE fields from `n` backwards to the root
- ENQUEUE(`nodes`, `q`) inserts in the queue `q` each node in the set `nodes`, in a position defined by the **search strategy**. Accordingly, a **different** implementation of ENQUEUE must be defined for each search strategy

A note on the representation of the state space

For some search problems the state space can be very large.
For instance, the state space of the 8-puzzle game has size $9!/2$.



Start State Goal State

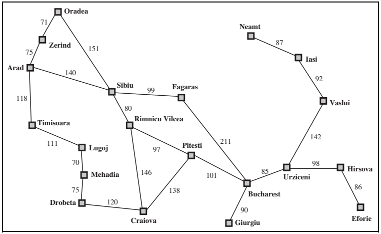
Fortunately, it is not necessary to store the whole state space in memory (e.g., represented as a graph): each state can indeed be obtained from the initial one through the successor function *SF* (remember that the tree-search algorithm iteratively constructs partial solutions as sequence of actions from the initial state).

In other words, as already pointed out previously, the initial state and *SF* **implicitly** define the state space.

A note on the representation of the state space

As a particular case, in problems like route finding in maps, defining the *SF* and the step cost function amounts to **explicitly** store the **whole** state space, i.e., the graph structure which corresponds to the map:

- the set of cities
- what pairs of cities are directly connected
- the cost of moving between any two adjacent cities



Measuring problem-solving performance

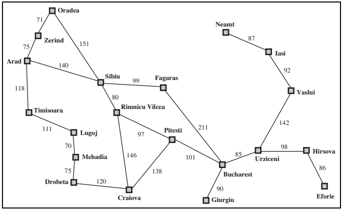
The performance of a tree-search algorithm can be evaluated according to two main criteria:

- **effectiveness**: how "good" is the solution found (if any)?
  - **completeness**: is the algorithm **guaranteed** to find a solution, when there is one?
  - **optimality**: when a solution is found, is its path cost **minimal**?
- **efficiency**: what is the processing cost of finding a solution (if any)? Formally, this property is called **computational complexity**:
  - **time** complexity: **how long** does it take to find a solution?
  - **space** complexity: **how much memory** is needed?

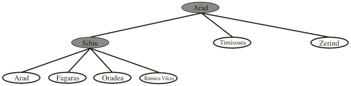
Often a **trade-off** between effectiveness and efficiency is required.

Search strategies

Search algorithms differ **only** in the criterion to choose one of the partial solutions to follow up at each step. An example:



current search tree:



which of the six partial solutions should one choose?

Two kinds of strategies exist, depending on the available information about which choice is "better" than another:

- no information: **uninformed** search strategies **must** be used
- some information: **informed** search strategies **can** be used

Uninformed search strategies

Rationale: in absence of any information about the “best” partial solution, **systematically** explore the space state.

Main strategies:

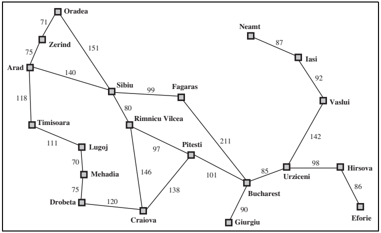
- breadth-first
- depth-first
- uniform-cost
- depth-limited
- iterative-deepening depth-first
- bidirectional

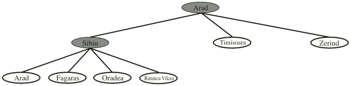
Breadth-first search (BFS)

BFS expands first the **shallowest** leaf node. If there is more than one leaf node at the lowest depth, one of them is **randomly** chosen.

This amounts to expand first all nodes at depth 0 (the root), then all nodes at depth 1, then all nodes at depth 2, and so on.

An example:

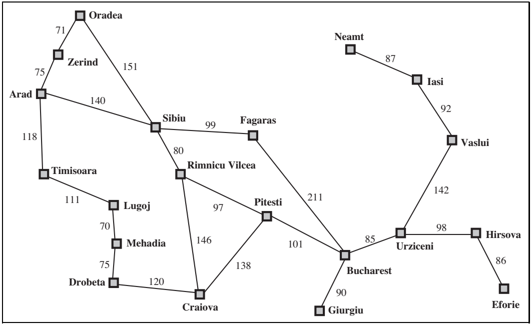




shallowest leaf nodes:
Timisoara, Zerind;
one of them is randomly chosen

Example of breadth-first search (1/6)

Route finding on maps: getting from Arad to Bucharest.



The nodes of the search tree will be numbered to to avoid ambiguities, since different nodes can be associated with the same state. The search steps are numbered according to the **general tree-search algorithm** shown above.

Example of breadth-first search (2/6)

1. root node: the initial state, Arad; fringe = { Arad }

2.1 the fringe is not empty

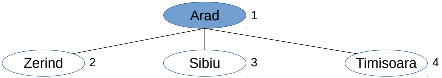
2.2 the **shallowest** node in the fringe (the root) is chosen (Arad)

2.3 Arad is not a goal state

Example of breadth-first search (3/6)

2.4 Arad is removed from the fringe and expanded, generating three nodes having **identical** depth

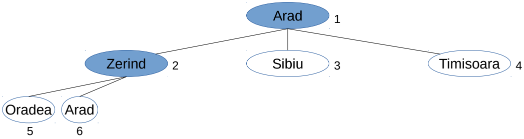
- the newly generated nodes must be inserted in the fringe (a queue) in the order in which they will be expanded by BFS; by definition, in BFS a newly expanded node has depth equal or higher than **all** the other nodes in the fringe; therefore, the newly expanded nodes are inserted **at the end** of the fringe
- since the newly generated nodes always have **identical** depth, they are inserted in the fringe in **any** order between themselves



Current fringe: { Zerind (2), Sibiu (3), Timisoara (4) }.

Example of breadth-first search (4/6)

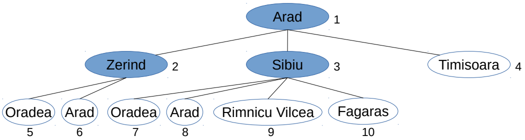
2.1 the fringe is not empty
2.2 the **first** node in the fringe is selected (it is guaranteed to be one of the shallowest leaf nodes)
2.3 the corresponding state, Zerind, is not the desired state
2.4 Zerind is removed from the fringe and expanded, generating two nodes that are inserted at the end of the fringe



Current fringe: { Sibiu (3), Timisoara (4), Oradea (5), Arad (6) }.

Example of breadth-first search (5/6)

2.1 the fringe is not empty
2.2 the **first** node in the fringe is selected
2.3 the corresponding state, Sibiu, is not the desired state
2.4 Sibiu is removed from the fringe and expanded, generating four nodes that are inserted at the end of the fringe



Current fringe: { Timisoara (4), Oradea (5), Arad (6), Oradea (7), Arad (8), Rimnicu Vilcea (9), Fagaras (10) }.

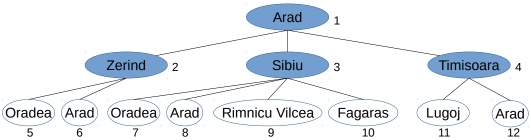
Example of breadth-first search (6/6)

2.1 the fringe is not empty

2.2 the **first** node in the fringe is selected

2.3 the corresponding state, Timisoara, is not the desired state

2.4 Timisoara is removed from the fringe and expanded, generating two nodes that are inserted at the end of the fringe

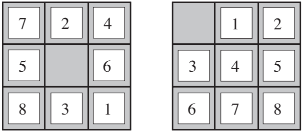


Current fringe: { Oradea (5), Arad (6), Oradea (7), Arad (8), Rimnicu Vilcea (9), Fagaras (10), Lugoj (11), Arad (12) }.

And so on.

Exercise

Apply the BFS algorithm to the 8-puzzle problem, considering the initial and goal states below, and **expanding** the first **four** nodes of the search tree (i.e., execute the first four iterations of the **general tree-search algorithm**).



Start State Goal State

Properties of breadth-first search

In terms of **effectiveness**, it can be easily shown that BFS is:

- **complete**: a solution is always found, if one exists
- **non-optimal**: it is not guaranteed that the solution with minimum path cost is found (if any), unless the path cost is a non-decreasing function of depth

Computational complexity can be evaluated as follows.

Computational complexity of algorithms

The execution time of a given algorithm depends on several factors not intrinsic to the algorithm itself: its implementation in a specific language, the hardware on which the program is executed, etc.

Time complexity is therefore evaluated not in terms of the execution time, but in terms of the number of "elementary operations" carried out by the algorithm, assuming that each of them can be executed in constant time.

Depending on the algorithm, "elementary operations" can be additions, multiplications, comparisons, iterations, etc.

The first step is therefore to identify what the "elementary operations" of a given algorithm are. For instance:

- sorting algorithms (selection sort, quick sort, etc.): comparison between a pair of values
- computing the digits of the representation of a number in a given basis: computing the quotient and remainder of a division

Computational complexity of algorithms

The number of elementary operations carried out by an algorithm depends either on the **value** or the **size** of its input, e.g.:

- to sort a sequence of numbers, the number of comparisons depends on the **size** of the sequence
- to computing the digits of the representation of a number in a given basis, the number of divisions depends on its **value**

Even if the number of elementary operations depends on more than one factor, for the sake of simplicity a single factor is considered and the others are kept constant.

Computational complexity of algorithms: an example

Consider the well-known **selection sort** algorithm, that sorts a sequence of values in increasing or decreasing order.

It can be described as follows:

1. find the minimum value in the sequence
2. swap it with the value in the first position
3. repeat the above steps for the remainder of the sequence, starting at the second position, then at the third one, up to the **penultimate** position

Computational complexity of algorithms: an example

A possible implementation of selection sort in C language, for sorting in non-decreasing order an array of integers:

```
void selection_sort (int a[], int length) {
   int i, j, ind_min, tmp;
   for (i = 0; i < length - 1; i++) {
      ind_min = i;
      for (j = i + 1; j < length; j++)
         if (a[j] < a[ind_min]) ind_min = j;
      if (ind_min != i) {
         tmp = a[i]; a[i] = a[ind_min]; a[ind_min] = tmp;
      }
   }
}
```

Computational complexity of algorithms: an example

Time complexity of selection sort can be evaluated as follows:

- each **comparison** in the nested loop can be considered as the **elementary operation**
- for a sequence of length n, the outer loop is repeated for $n-1$ times
- at iteration k ($k = 1, \ldots, n-1$), $n-k$ comparisons are made to find the lowest element, starting from the k-th position, then a swap is possibly made.

The **exact** number of comparisons is therefore given by:

$$(n-1) + (n-2) + \ldots + 2 + 1 = \frac{n(n-1)}{2}$$

Time complexity depends therefore on the **size** of the input, i.e., the sequence length n. Therefore, all problem instances of a given size (sequences of a given length) have the same time complexity.

Computational complexity of algorithms

Let n denote the value of the factor which the number of elementary operations carried out by an algorithm depends on (e.g., the length of a sequence to be sorted, or the value of a number to be represented in a given basis), and $f(n)$ the corresponding number of of elementary operations.

For some algorithms evaluating $f(n)$ can be difficult, especially if it depends on the problem **instance**, i.e., on the input value.

Time complexity is thus evaluated only for some particular cases, usually:

- **worst-case** complexity, corresponding to the instances that require the **highest** number of elementary operations
- **average-case** complexity: average number of elementary operations over **all** possible problem istances
- **best-case** complexity, corresponding to the instances that require the **lowest** number of elementary operations

Computational complexity of algorithms

To evaluate and compare algorithms it useful to consider their **asymptotic** time complexity, i.e., the behaviour of $f(n)$ as $n \to \infty$.

To this aim an **upper bound** $g(n)$ of $f(n)$ is determined using asymptotic analysis, known as "big $\mathcal{O}$ notation".

A function $f(n)$ is said to be $\mathcal{O}(g(n))$ ("order g"), if there exists some $n_0 > 0$ and $c > 0$ such that $f(n) \leq c \times g(n)$ for each $n \geq n_0$. Of course, the **tightest** upper bound $g(n)$ is of interest.

For instance, it is easy to see that any polynomial of degree p, $a_p n^p + a_{p-1} n^{p-1} + \ldots + a_1 p + a_0$, is $\mathcal{O}(n^p)$.

As an example, this implies that the time complexity of selection sort, given by $\frac{n(n-1)}{2}$, is $\mathcal{O}(n^2)$.

Computational complexity of algorithms

Well-known categories of (increasing) asymptotic complexity are the following:

- $\mathcal{O}(1)$: constant time algorithms: their execution time is identical for all problem instances
- $\mathcal{O}(logn)$: logarithmic time (e.g., binary search in sorted sequences)
- $\mathcal{O}(n)$: linear time (e.g., sequential search)
- $\mathcal{O}(n^p)$, for a given integer p: polynomial time (e.g., selection sort, with $p = 2$)
- $\mathcal{O}(k^n)$, for a given $k > 1$: exponential time (e.g., the simplex algorithm in linear programming)

Computational complexity of search algorithms

Getting back to the **general tree-search algorithm**:

- elementary operation: **generation of a new node** during the expansion of a leaf node (step 2.4)
- data to be stored in memory: **nodes** of the search tree (a constant amount of memory for each node can be assumed)

Computational complexity can therefore be evaluated as follows:

- worst-case **time** complexity: the highest number of nodes that are **generated** before a solution is found (if any)
- worst-case **space** complexity: the highest number of nodes that have to be **simultaneously** stored in memory

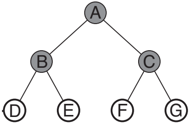
Breadth-first search: computational complexity

In the specific case of BFS, it is not difficult to see that computational complexity depends on two main factors:

- the number of successors of each node of the search tree
- the depth d of the **shallowest** solution, which is the one found by BFS

Since different nodes can have a different number of successors (see, e.g., 8-puzzle and route finding on maps), to simplify the evaluation of computational complexity, a **constant** number of successors b, named **branching factor**, is considered.

For instance, for $b = 2$ we have a **binary tree**:



Breadth-first search: computational complexity

For a **fixed** branching factor, computational complexity can be evaluated as a function of d only.

Time complexity: in the **worst case** the goal state is in the **last** node that is selected in step 2.2 to be expanded, among all the ones at depth d. This means that all the other nodes at depth d are expanded before.

The number of generated nodes can be computed by evaluating the number of nodes that are generated at each depth:

Depth	**Number of generated nodes**
0	1 (root node)
1	b
2	b^2
3	b^3
$\ldots$	$\ldots$
d	b^d
$d+1$	$b^{d+1} - b$
Total:	$1 + b + b^2 + b^3 + \ldots + b^d + (b^{d+1} - b)$

Breadth-first search: computational complexity

To evaluate **space** complexity it suffices to notice that **all** generated nodes must remain in memory until a solution is found. It follows that space complexity equals time complexity.

The worst case time and space complexity of BFS, for a search tree with a given branching factor b and shallowest solution at depth d, are therefore given by:

$$1 + b + b^2 + b^3 + \ldots + b^d + (b^{d+1} - b) \ .$$

It easily follows that the asymptotic complexity of BFS is $\mathcal{O}(b^{d+1})$, i.e., it is **exponential** with respect to the depth of the shallowest solution.

In general, an exponential computational complexity denotes a very low efficiency.

Breadth-first search: computational complexity

As an example of what an exponential complexity means, consider a search problem with the following features:

- branching factor $b = 10$
- time for generating one node: 10^{-4} s
- storage required for a single node: $1,000$ bytes

Worst-case time and space complexity of BFS, as a function of the depth d of the shallowest solution:

Depth	Nodes	Time	Memory
2	$1,100$	0.11 sec.	1 megabyte
4	$111,100$	11 sec.	106 megabytes
6	10^7	0.19 minutes	10 gigabytes
8	10^9	31 hours	1 terabyte
10	10^{11}	129 days	101 terabytes
12	10^{13}	35 years	10 petabytes
14	10^{15}	$3,523$ years	1 exabyte

Properties of breadth-first search

To sum up, BFS exhibits the following properties:

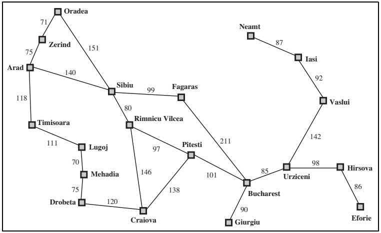
- it is **complete**: a solution is always found, if one exists
- it is **non-optimal**: it is not guaranteed that the solution with minimum path cost is found (if any), unless the path cost is a non-decreasing function of depth
- it has an **exponential time and space complexity** with respect to the depth of the shallowest solution

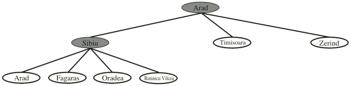
Depth-first search (DFS)

Contrary to BFS, DFS expands first the **deepest** leaf node (in case there are several such nodes, a random choice is made).

This amounts to explore first one **whole** path; if no solution is found, another **whole** path is explored, and so on.

An example:

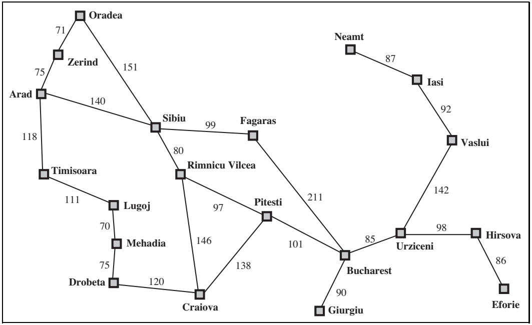




deepest leaf nodes: Arad, Fagaras, Oradea, Rimnicu Vilcea;
one of them is randomly chosen

Example of depth-first search (1/5)

Route finding on maps: getting from Arad to Bucharest.



Example of depth-first search (2/5)

1. root node: the initial state, Arad; fringe = { Arad }

2.1 the fringe is not empty

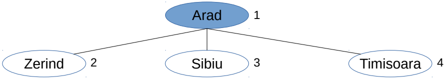
2.2 the **deepest** node in the fringe (the root) is chosen (Arad)

2.3 Arad is not a goal state

Example of depth-first search (3/5)

2.4 Arad is removed from the fringe and expanded, generating three nodes having **identical** depth

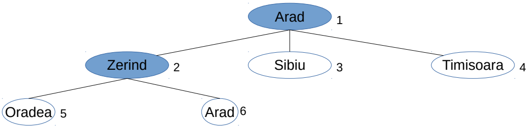
- the newly generated nodes must be inserted in the fringe (a queue) in the order in which they will be expanded by DFS; by definition, in DFS a newly expanded node has depth equal or higher than **all** the other nodes in the fringe; therefore, the newly expanded nodes are inserted **at the top** of the fringe
- since the newly generated nodes always have identical depth, they are inserted in the fringe in **any** order between themselves



Current fringe: { Zerind (2), Sibiu (3), Timisoara (4) }.

Example of depth-first search (4/5)

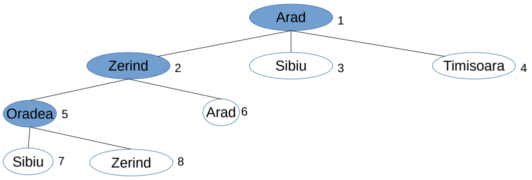
2.1 the fringe is not empty
2.2 the **first** node in the fringe is selected (it is guaranteed to be one of the deepest leaf nodes)
2.3 the corresponding state, Zerind, is not the desired state
2.4 Zerind is removed from the fringe and expanded, generating two nodes that are inserted at the top of the fringe



Current fringe: { Oradea (5), Arad (6), Sibiu (3), Timisoara (4) }.

Example of depth-first search (5/5)

2.1 the fringe is not empty
2.2 the **first** node in the fringe is selected
2.3 the corresponding state, Oradea, is not the desired state
2.4 Oradea is removed from the fringe and expanded, generating two nodes that are inserted at the top of the fringe

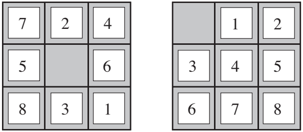


Current fringe:
{ Sibiu (7), Zerind (8), Arad (6), Sibiu (3), Timisoara (4) }.

And so on.

Exercise

Apply the DFS algorithm to the 8-puzzle problem, considering the initial and goal states below, and **expanding** the first **four** nodes of the search tree (i.e., execute the first four iterations of the **general tree-search algorithm**).



Start State Goal State

Some remarks on depth-first search

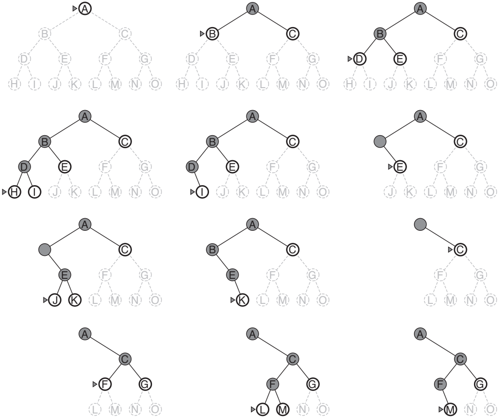
A drawback of DFS is that it can get stuck going down along very long paths. **Infinite** paths can also occur, due for instance to loops like: Arad $\rightarrow$ Zerind $\rightarrow$ Arad $\rightarrow$ Zerind ... (loops can be easily avoided – see later).

On the other hand, DFS has modest memory requirements:

- if **all** the paths starting from a given node have been fully explored (if they are not infinite) and no solution has been found, the sub-tree having such a node as the root can be **removed** from memory
- it follows that only a **single** path form the root to a leaf node needs to be stored in memory during the search, together with the unexpanded sibling nodes for each node on that path

Example of depth-first search

An example for a **binary** search tree, assuming that each path has maximum depth 3, and that node `M` contains a goal state. Shaded nodes are the ones not yet generated. Among several nodes at the same depth, the left-most one is chosen for expansion in this example.



Depth-first search: computational complexity

Computational complexity of DFS can be evaluated by assuming that:

- all nodes have the same number of successors b (branching factor)
- all solutions have the same depth m
- m is also the **maximum** depth of the search tree (worst case), assuming loops are avoided

Moreover, in the worst case the goal state is in the **last** path that is explored by DFS.

To compute **time** complexity, note that the above assumptions imply that **all** nodes up to depth m are generated before the solution is found.

To compute **space** complexity, remember that **only a single path** from the root to a leaf node, along with the remaining unexpanded sibling nodes for each node in the path, must be stored (see the example above).

Depth-first search: computational complexity

Under the above assumptions, the computational complexity of DFS as a function of the solution depth m is given by:

Depth	Time complexity **N. of generated nodes**	Space complexity **N. of stored nodes**
0	1 (root node)	1 (root node)
1	b	b
2	b^2	b
...	...	...
m	b^m	b
Total:	$1 + b + \ldots + b^m = \mathcal{O}(b^m)$	$1 + mb = \mathcal{O}(m)$

Time complexity is therefore exponential with respect to the solution depth m, as that of BFS, but space complexity is **linear**.

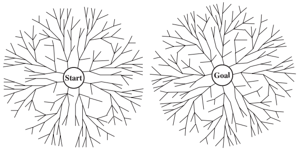
Properties of depth-first search

DFS exhibits the following properties:

- it is **complete**, unless there are infinite paths
- it is **non-optimal**: a deeper, suboptimal solution can be found along a path that is explored before another one containing the optimal solution at smaller depth
- it has an **exponential time complexity** with respect to the solution depth, but only a **linear space complexity**

Other strategies

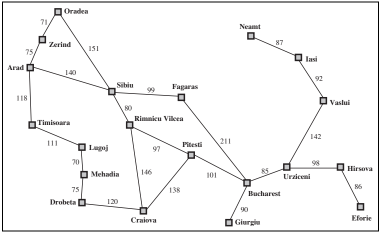
- **Uniform-cost**: expands the leaf node with the lowest path cost (it is both **optimal** and **complete**)
- **Depth-limited**: depth-first search with a predefined depth limit (avoids infinite paths, but is **not complete**)
- **Iterative-deepening depth-first**: repeated depth-limited search with depth limit $1, 2, 3, \ldots$, until a solution is found (it avoids infinite paths, and is **complete**)
- **Bidirectional**: simultaneously searching **forward** from the initial state and **backwards** from the goal state, with alternate node expansions, until the two searches meet, i.e., a common state is found in their frontiers (requires reversible actions)



Avoiding repeated states

Search algorithms may waste time by expanding **different** nodes associated with the **same** state. This may happen, e.g., when:

- actions are **reversible**, which allows **loops** like: Arad → Zerind → Arad → Zerind . . .
- **different** paths can lead to the **same** state, e.g.: Arad → Sibiu, and Arad → Zerind → Oradea → Sibiu
- **cyclical paths** exist (loops are a particular case), e.g.: Arad → Zerind → Oradea → Sibiu → Arad



Avoiding repeated states

Four main solutions of increasing complexity can be adopted.
When a node `n` is being expanded:

1. if reversible actions exist, discard the newly generated node containing the same state of the **parent** node of `n` (this avoids loops)
2. discard all children nodes containing states already present **in the same path** from the root to `n` (this avoids cyclical paths)
3. discard all children nodes containing states already present **in the current search tree** (ineffective for DFS, which does not store all generates nodes)
4. discard all children nodes containing states **previously** generated, even if not present in the current search tree

Avoiding repeated states

The above solutions require to compare every newly generated node with some other nodes, which can affect computational complexity:

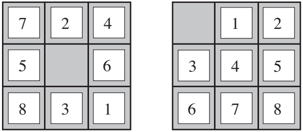
- solution 1 requires a single comparison
- solution 2 requires a number of comparisons equal to the depth of the parent node `n`
- solution 3 requires a comparison with **all** the nodes in the **current** search tree: its **time complexity** is **exponential** for strategies with exponential space complexity (like BFS)
- solution 4 requires to store the states of **all previously generated** nodes (this can significantly increase the space complexity of DFS)

A careful implementation of strategies 3 and 4 would remove the repeated state with the **highest** path cost, to avoid ruling out the optimal solution.

Effectiveness of uninformed search: an example

One may think that the high computational complexity of uninformed search strategies is an issue only for real-world problems, not for toy ones.

Consider again 8-puzzle, apparently a very simple toy problem:



Start State Goal State

How long does it take to solve it using, e.g., BFS?

Effectiveness of uninformed search: an example

Some facts about 8-puzzle:

- the state space contains $9! = 362,880$ distinct states (only $9!/2 = 181,440$ are reachable from any given initial state)
- it can be shown that the average solution depth (over all possible pairs of initial and goal states) is about 22
- the average branching factor b (over all possible states) is about 3 (note that from each state 2 to 4 actions can be performed)

How many nodes does BFS generate and store, when the shallowest solution has depth $d = 22$ (i.e., in the **average** case)?

Remember that the worst-case time and space complexity of BFS is $\mathcal{O}(b^{d+1})$, which for $d = 22$ amounts to $3^{23} \approx 9 \times 10^{10}$...

For instance, taking into account that representing a state requires at least $\lceil \log_2 9! \rceil = 19$ bits, storing 3^{23} states requires about $19 \times 9 \times 10^{10}$ bits, i.e., more than 200 GB...

Exercise

1. Implement the **general tree-search algorithm**, and the related data structures, in a programming language of your choice
2. Implement the **additional**, specific functions for breadth-first, depth-first and uniform-cost search
3. Implement the **additional**, specific data structures and functions for the 8-puzzle problem, and the route finding problem in the Romania map
4. Run the above search algorithms on specific problem instances, and evaluate the number of generated nodes and the maximum number of nodes simultaneously stored in memory

Informed search

Uninformed search is based on systematically exploring the search space, and does not exploit any information (if any) about what nodes are more “promising” than others towards the solution.

When some knowledge is available, it can be exploited to improve the effectiveness and the efficiency of tree search.

The main idea is to use the available **problem-specific** knowledge to identify the **“best”** node to expand at each step of the general tree-search algorithm, instead of using uninformed criteria like expanding the shallowest or deepest node.

This general approach is named **best-first search**.

Best-first search

Best-first search is based on quantitatively evaluating how "promising" a given node `n` is towards a solution, through a suitable **node evaluation function** $f(\mathtt{n})$ (conventionally, lower values of f correspond to "better" nodes).

Different definitions of $f(\mathtt{n})$ lead to different, specific best-first search strategies, for instance:

- greedy search
- A*-search and its many variants (iterative-deepening A*, memory-bounded A*, etc.)

Best-first search

Once a suitable $f(\mathtt{n})$ (i.e., a specific best-first search strategy) has been defined, the corresponding search algorithm can be implemented using **the same general tree-search algorithm** presented above.

Best-first search can be easily implemented by sorting nodes in the fringe for increasing values of $f(\mathtt{n})$: this allows the node `n` with the lowest $f(\mathtt{n})$ (the "best" node) to be selected for expansion at each iteration.

Best-first search

To define $f(\mathtt{n})$, a very useful information is the cost of the actions that will lead from any given node $\mathtt{n}$ to a goal state.

Although in non-trivial problems the **exact** cost is usually unknown, often an **estimate** can be easily computed.

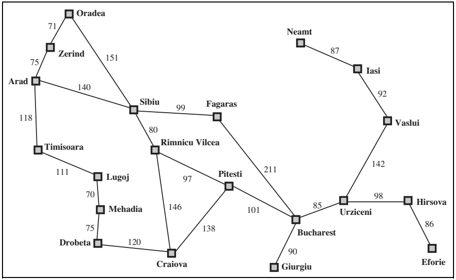
The estimated cost as a function of the nodes is formalized as a function $h(\mathtt{n})$. Note that, by definition, $h(\mathtt{n}) = 0$ if $\mathtt{n}$ contains a goal state (this is the only case in which the cost is **exactly** known).

For historical reasons $h(\mathtt{n})$ is named **heuristic function**, and search strategies based on it are named **heuristic search**.

Heuristic search is one of the early achievements of AI (dating back to the 1950's), but is still widely used in real-world problems and investigated by researchers in AI.

Heuristic functions: an example

Consider the problem of route finding in maps, e.g., finding the **shortest** route form Arad to Bucharest using the information on the map below:



Since the goal is to find the shortest route, the cost of the actions is evaluated as the route length (e.g., in Km). Defining an heuristic function for this problem amounts therefore to estimating the **distance** between any given city and the destination (in the problem instance above, Bucharest).

Heuristic functions: an example

An **easy to compute** estimate for this kind of problem is the **straight-line distance** (e.g., it can be easily computed from the geographical coordinates of each city).

If the destination (goal state) is Bucharest, the heuristic function $h(\mathtt{n})$ can therefore be defined as the straight-line distance from the city (state) of node `n` to Bucharest.

The values of $h(\mathtt{n})$, considering Bucharest as the destination, are shown on the right (they will be used later on).

Arad	366	**Mehadia**	241
Bucharest	0	**Neamt**	234
Craiova	160	**Oradea**	380
Drobeta	242	**Pitesti**	100
Eforie	161	**Rimnicu Vilcea**	193
Fagaras	176	**Sibiu**	253
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Iasi	226	**Vaslui**	199
Lugoj	244	**Zerind**	374

Greedy best-first search

This is the simplest best-first search strategy: expanding the node that **appears** to be the **closest** to the solution.

Since usually the exact cost is unknown, this strategy is implemented using the estimated cost, i.e., the heuristic function $h(\mathtt{n})$.

Accordingly, the node evaluation function is simply defined as:

$$f(\mathtt{n}) = h(\mathtt{n})$$

This strategy is called "greedy" since it favours the partial solution that **appears** to be the closest to the goal state (since $h(\mathtt{n})$ is only an estimate), but, as it will become more clear later, this is not an optimal choice.

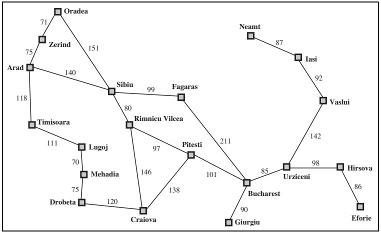
Greedy best-first search: an example

Consider again the problem of finding the shortest route from Arad to Bucharest, using the straight-line distance heuristic.

In the following, the search tree built by the greedy search strategy, until a solution is found, is shown, including the value of $f(\mathtt{n})$ for each node. The node selected for expansion is highlighted by an arrow.

Remember that using the **general tree-search algorithm** a solution is found when a node containing a goal state is selected to be **expanded**, **not** when it is **generated** by the expansion of its parent node.

Greedy best-first search: an example



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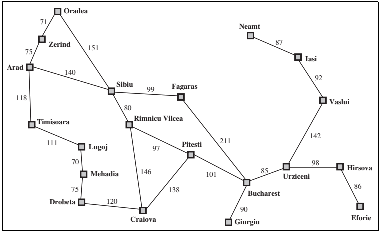
Lugoj	244	**Zerind**	374

(a) The initial state

Greedy best-first search: an example



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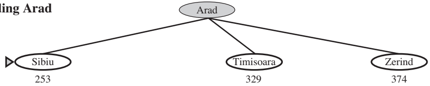
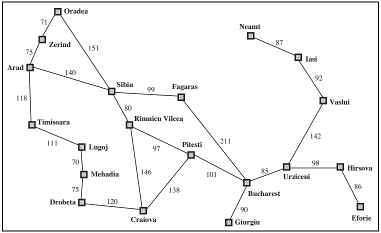
Iasi	226	**Vaslui**	199
Lugoj	244	**Zerind**	374

(b) After expanding Arad



Greedy best-first search: an example



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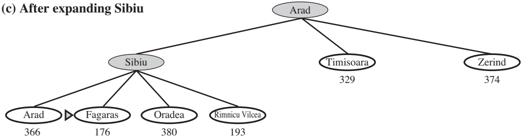
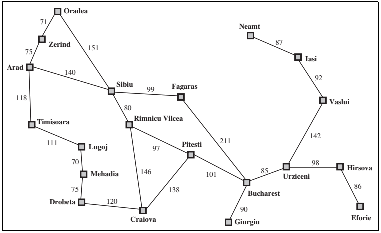
Hirsova	151	**Urziceni**	80
Iasi	226	**Vaslui**	199
Lugoj	244	**Zerind**	374

(c) After expanding Sibiu



Greedy best-first search: an example



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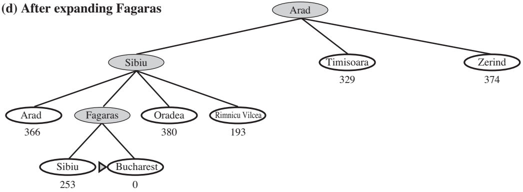
(d) After expanding Fagaras



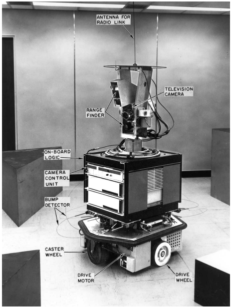
Properties of greedy best-first search

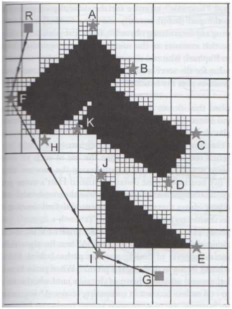
It can be shown that greedy best-first search exhibits the following properties:

- it is **complete**, unless there are infinite paths (including trivial loops)
- it is **non-optimal**: for instance, carefully looking at the above example it can be seen that a shorter route between Arad and Bucharest exists (through Sibiu and Rimnicu Vilcea), than the one found by greedy search
- worst-case **time and space complexity** is **exponential** in the depth of the shallowest solution m, for a given branching factor b: $\mathcal{O}(b^m)$

A* search

A* is the most relevant best-first search strategy. It was devised in the 1960s for robot navigation tasks.





Many variants of A* have been proposed since then to tune the trade-off between its effectiveness and efficiency.

A* search

Greedy search chooses for expansion the node n which appears closest to the goal state, i.e., such that the estimated cost of the actions from n to a goal state is minimum. However, it disregards the cost of the actions from the root to n.

A* uses instead an estimate of the **total** cost of the action sequence from the root to a goal state through n, defined as the sum of the path cost of n (which is **exactly** known) and the estimated cost from n to the solution.

The corresponding node evaluation function is defined as:

$$f(\mathtt{n}) = \underbrace{g(\mathtt{n})}_{\text{path cost from root to } \mathtt{n}} + \underbrace{h(\mathtt{n})}_{\text{estimated cost from } \mathtt{n} \text{ to the solution}}$$

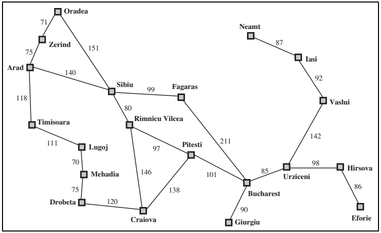
A* search: an example

In the following the search tree built by A* is shown for the same problem of the previous example (Arad-Bucharest, using the straight-line distance heuristic). The value of $f(\mathtt{n}) = g(\mathtt{n}) + h(\mathtt{n})$ is also shown for each node.

Note that after the fourth iteration (the expansion of the node containing the state Fagaras) a leaf node containing the goal state Bucharest is generated. However, it is not selected for expansion at the next iteration (and thus a solution is not found yet), since it is not the node with the minimum value of $f(\mathtt{n})$.

During the expansion of the node containing the state Pitesti in the fifth iteration, a **different** node containing the goal state Bucharest is generated. This latter node is selected for expansion at the next iteration, since it has the minimum value of $f(\mathtt{n})$, and since it contains a goal state, a solution is found and A* terminates.

A* search: an example



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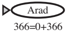
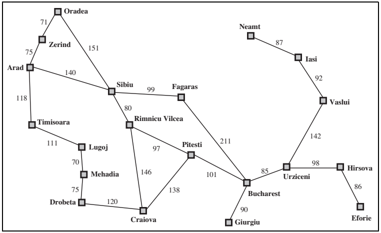
Iasi	226	**Vaslui**	199
Lugoj	244	**Zerind**	374

(a) The initial state



A* search: an example



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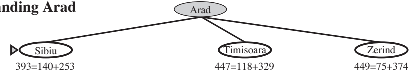
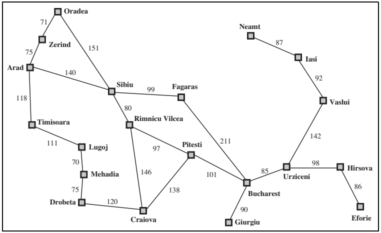
Iasi	226	**Vaslui**	199
Lugoj	244	**Zerind**	374

(b) After expanding Arad



A* search: an example



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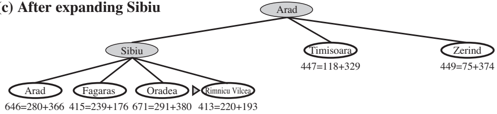
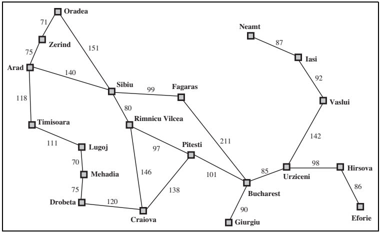
Lugoj	244	**Zerind**	374

(c) After expanding Sibiu



A* search: an example



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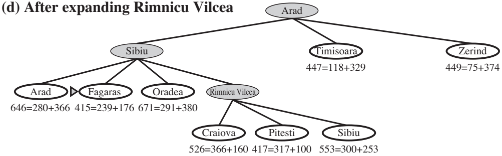
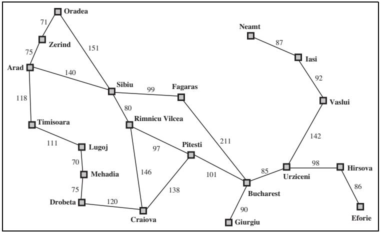
(d) After expanding Rimnicu Vilcea



A* search: an example



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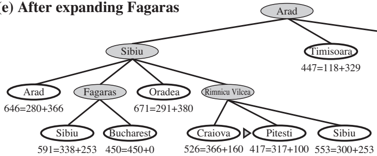
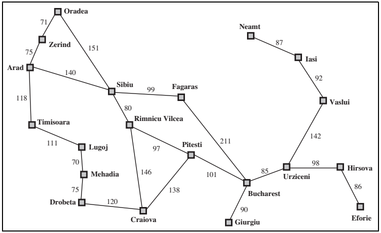
Lugoj	244	**Zerind**	374

(e) After expanding Fagaras



A* search: an example



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Giurgiu	77	**Timisoara**	329
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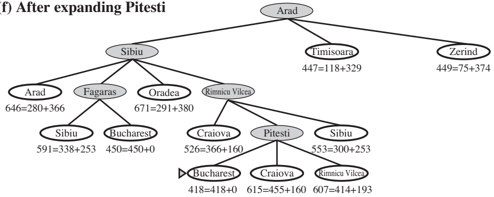
Iasi	226	**Vaslui**	199
Lugoj	244	**Zerind**	374

(f) After expanding Pitesti



Properties of A* search

It can be shown that A* search exhibits the following properties:

- it is **optimal** (the proof is given in the following), provided that the heuristic is **admissible**, i.e., never overestimates the cost to the solution (e.g., the straight-line distance is an admissible heuristic for route finding in maps)
- it is **complete**, and is also **optimally efficient** (i.e., it expands the **minimum** number of nodes) for any admissible heuristic, among algorithms that extend search paths from the root
- worst-case **time and space complexity** are **exponential** in the depth of the shallowest solution m, for a given branching factor b: $\mathcal{O}(b^m)$; nevertheless, A* is often **much more efficient** (i.e., it generates a much smaller number of nodes) than other uninformed and informed search strategies

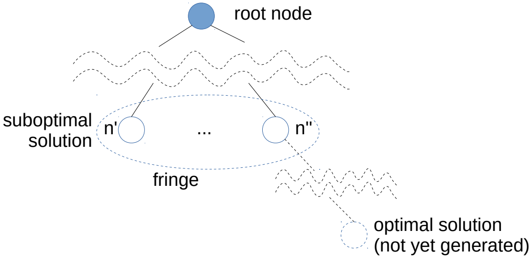
Proof of A* optimality

Assume the fringe contains one leaf node $\mathtt{n}'$ with a suboptimal goal state, and no leaf node with an optimal goal state.

Can A* ever select $\mathtt{n}'$ to be expanded, thus returning it as a **suboptimal** solution?

First, note that some leaf node $\mathtt{n}'' \neq \mathtt{n}$ in the path toward an optimal solution **must** exist in the fringe.

We have to consider therefore the following scenario:



Proof of A* optimality

Denoting with C^* the cost of an optimal solution, the above assumptions imply:

1. $h(\mathtt{n}') = 0$ ($\mathtt{n}'$ contains a goal state)
2. $f(\mathtt{n}') = g(\mathtt{n}') + h(\mathtt{n}') = g(\mathtt{n}') > C^*$
   ($\mathtt{n}'$ contains a sub-optimal goal state)
3. $f(\mathtt{n}'') = g(\mathtt{n}'') + h(\mathtt{n}'') \leq C^*$ ($\mathtt{n}''$ is in the path toward an optimal solution, and h is admissible, thus $f(\mathtt{n}'')$ does not overestimate the cost of any solution reachable through $\mathtt{n}''$)

In turn, expressions 2 and 3 imply:

$$f(\mathtt{n}') > C^* \geq f(\mathtt{n}'')$$

This means that $\mathtt{n}'$ **cannot** be selected for expansion, and thus A* **cannot** return a suboptimal solution.

Improving A* search

- "Good" heuristics (discussed later) can reduce time and memory requirements, especially with respect to uninformed search
- However, in many practical problems even A* is infeasible: memory requirements are the main drawback. Alternative approaches have been devised:
  - using **non-optimal** A* variants that find suboptimal solutions quickly
  - using A* variants with reduced memory requirements and a small increase in execution time, but still optimal

Defining heuristic functions

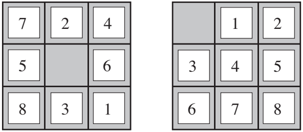
Intuitively, the more accurate is the estimate of the cost to the solution from a given node provided by the heuristic function, the more efficient a best-first algorithm is.

Defining a **good** (i.e., accurate) heuristic is therefore crucial for informed search. Moreover, heuristics have to be **admissible** to guarantee the optimality of A*.

Defining heuristic functions: examples

We have seen that a possible heuristic for route finding in maps is the straight-line distance.

Consider now the 8-puzzle problem. Remember that about 9×10^{10} nodes are generated on average by breadth-first (uninformed) search when the solution is at the average depth of 22: therefore a good heuristic can be of great practical help also in this toy problem.



Start State   Goal State

As an **exercise**, try to devise admissible heuristic functions for the 8-puzzle problem.

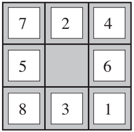
Defining heuristic functions

Well-known admissible heuristics for k-puzzle are the following:

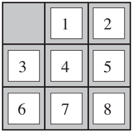
- number of misplaced tiles (in the following, $h_1(\mathtt{n})$)
- sum of the "distances" of each tile from its goal position (**city block** or **Manhattan** distance, $h_2(\mathtt{n})$)

For instance, the value of h_1 and h_2 for the start state below on the left, with respect to the goal state on the right, is given by:

- $h_1(\text{start state})$: 8 (all 8 tiles are misplaced)
- $h_2(\text{start state})$: $3+1+2+2+2+3+3+2=18$ (tiles 1 to 8)



Start State



Goal State

Defining or choosing heuristic functions

For some problems it may be not straightforward to define a heuristic function. In that case a general approach is to set $h(\mathtt{n})$ equal to the **exact** cost of a **relaxed** version of the problem at hand, which may be easy to compute.

Some examples:

- k-**puzzle**: by relaxing the constraint that tiles can move only to a **free** adjacent square, and allowing them to move to **any adjacent** square, one obtains $h_2(\mathtt{n})$ (see above)
- k-**puzzle**: similarly, allowing tiles to move to **any** square (even non-adjacent and occupied ones), one obtains $h_1(\mathtt{n})$
- **route finding in maps**: by relaxing the constraint that an adjacent city can be reached only through the corresponding route, and allowing one to move straight to it, one obtains the straight-line distance heuristic

Defining or choosing heuristic functions

On the other hand, for some problems it can be possible to define several admissible heuristics $h_1, \ldots, h_p$ (e.g., h_1 and h_2 for 8-puzzle).

In this case one can choose or define a single heuristic h which **dominates** all the other ones, i.e.:

$$\textbf{for each } \text{node } \mathtt{n},\ h(\mathtt{n}) \geq h_i(\mathtt{n}), i = 1, \ldots, p.$$

It is easy to see that such a heuristic is admissible, and provides a more accurate estimate of the cost to the solution than $h_1, \ldots, h_p$.

To this aim, h can be defined as follows:

- if there is a dominating heuristic among $h_1, \ldots, h_p$, choose it as the heuristic for the problem at hand
- otherwise, for a given node n use the following heuristic:

$$h(\mathtt{n}) = \max\{h_1(\mathtt{n}), \ldots, h_p(\mathtt{n})\},$$

  which dominates by definition $h_1, \ldots, h_p$

Evaluating heuristic functions

To evaluate the quality of heuristic functions the concept of **effective branching factor** (denoted as b^*) is used:

- let N be the number of nodes generated by A* for a given problem, and d be the depth of the (optimal) solution
- b^* is defined as the branching factor of a **uniform** tree of depth d containing N nodes, which is the solution of the equation:

$$N = 1 + b^* + (b^*)^2 + \cdots + (b^*)^d$$

The **lower** the value of b^*, the better the heuristic.

Since b^* depends on the problem **instance**, it is usually evaluated **empirically** as the **average** over a set of instances.

Evaluating heuristic functions

As an example, the table below reports the results of an empirical evaluation of the effective branching factor of heuristics h_1 and h_2 for the 8-puzzle (used in A*), and, for comparison, of one of the most efficient uninformed search strategies, iterative-deepening depth-first search (IDS).

The comparison is made on 600 randomly generated problem instances with solution depth $d = 4, 8, \ldots, 24$ (100 instances for each depth value). The symbol – means that IDS could not terminate due to memory overflow. It is clear that h_2 is significantly better than h_1, and that uninformed search is unfeasible even for 8-puzzle.

	search cost (expanded nodes)			effective branching factor		
d	IDS	A* (h_1)	A* (h_2)	IDS	A* (h_1)	A* (h_2)
4	112	13	12	2.87	1.48	1.45
8	6,384	39	25	2.80	1.33	1.24
12	3,644,035	227	73	2.78	1.42	1.24
16	–	1,301	211	–	1.45	1.25
20	–	7,276	676	–	1.47	1.27
24	–	39,135	1,641	–	1.48	1.26