University of Cagliari

MSc programs in: Computer Engineering, Cybersecurity and Artificial Intelligence – Electronic Engineering – Internet Engineering

Artificial Intelligence

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Exercises on Bayesian networks

1. Consider the following Boolean random variables related to the state of a car: *Battery* (it equals $false$ if the battery is dead), *Fuel* (it equals $false$ if the fuel tank is empty), *Ignition* (it equals *true* if the ignition system works), $Moves$ (it equals *true* if the car moves after one tries to start the engine), *Radio* (it equals *true* if the radio works when one tries to switch it on), $Starts$ (it equals *true* if the engine fires when one tries to start it).
    (a) Define a proper set of causal relations between the events corresponding to the above random variables, and write the corresponding expression of their joint probability distribution function using the chain rule.
    (b) Make suitable conditional independence assumptions, *clearly motivating them*, and show how they modify the expression of the chain rule.
    (c) Draw a Bayesian network representing the joint distribution obtained from point 1b.
    (d) How many probability values need to be estimated to define the distributions associated to the nodes of your Bayesian network? Can some of these probability values be set based on *a priori* causal knowledge about the corresponding events?
2. Two astronomers, in different parts of the world, look at the same region of the sky using their telescopes and count the number of stars they see. Their counts may be inaccurate for several reasons, including the fact that their telescopes can occasionally be out of focus.
    (a) Define a set of random variables to describe the above domain.
    (b) Define a proper set of causal relations between the considered variables.
    (c) Draw a Bayesian network to represent their joint distribution, making suitable conditional independence assumptions.
    (d) Write the corresponding expression of the joint distribution. Do you see any reasonable constraint that some of the resulting distribution should satisfy, based on *a priori* causal knowledge about the corresponding events?
3. Mary's car has an alarm that sounds when a motion sensor detects someone entering it. The alarm and the sensor are powered by two distinct batteries, which can be occasionally dead.
    (a) Define a suitable set of random variables to represent this domain.
    (b) Order these variables according to proper causal relations.
    (c) Draw a Bayesian network to represent their joint distribution, according to the order defined in point 3b, making suitable conditional independence assumptions.
    (d) Write the corresponding expression of the joint distribution.

4. Headache and fever are among the symptoms of several health problems, including influenza and food poisoning.

   (a) Represent the above knowledge as a Bayesian network, defining suitable random variables and a proper order between them. Clearly explain the conditional independence assumptions you make.

   (b) Write the corresponding expression of the joint distribution.

   (c) Derive an expression of the probability that a person who is suffering from headache has caught influenza.

5. In a nuclear power station an alarm sounds and warning lights flash in the control room, when a sensor detects that the temperature of the core exceeds a given threshold. The sensor measurement may be incorrect (very rarely), resulting in false positive or false negative detections; this is less unlikely to happen, when the external temperature gets too high. Occasionally, also the alarm and the warning lights can fail; to limit joint failures, they are implemented as physically separated systems.

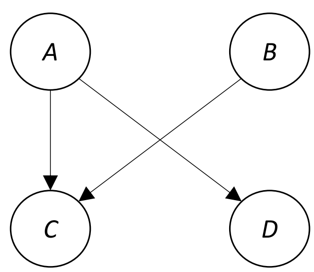
   (a) Represent the above domain as a Bayesian network, defining suitable random variables and a proper order between them. Clearly explain the conditional independence assumptions you make.

   (b) Write the corresponding expression of the joint distribution.

   (c) Derive an expression of the probability of a core overheating, when warning lights are flashing in the control room.

6. Draw the Bayesian network that represents the joint distribution function of four random variables A, B, C and D, assuming that the causal relations between them are $D \to C \to B \to A$ (i.e., D is the "root cause"), and that *no* conditional independence assumption can be made. Write the full joint distribution function using the chain rule corresponding to that Bayesian network. Assuming all four variables are Boolean, how many probability values need to be estimated to specify the distribution functions associated with the Bayesian network?

7. Assume that the Bayesian network below has been obtained by considering the following order between its random variables, from root causes to end effects: A, B, C, D:



   (a) Write the corresponding expression of the full joint distribution function using the chain rule.

   (b) What conditional independence assumptions does the above Bayesian network encode?

   (c) Assuming the variables are all Boolean, derive an expression for the conditional probability

$$P(A = true|C = true, D = true)$$

   (d) Sketch the procedure for estimating the same probability above from the Bayesian network using the *rejection sampling* algorithm, hypothesising at each step the sampled value of the corresponding random variable.

Solution

1. (a) The state of the battery and of the fuel tank can be considered as the "root causes". The battery state directly affects the working of the radio and of the ignition system. In turn, the states of the fuel tank and of the ignition system directly determine whether the engine fires or not. Finally, the state of the engine directly determines whether the car moves.

   Accordingly, the considered random variables can be sorted from the "root causes" to the "end effects" as follows: $\{Fuel, Battery\} \to \{Radio, Ignition\} \to Starts \to Moves$. Note that the order between a subset of variables inside curly brackets is not relevant. In the following, this order will be considered: $Fuel, Battery, Radio, Ignition, Starts, Moves$. Denoting the random variables with their initial letters only, for the sake of brevity, the corresponding expression of their joint distribution function using the chain rule is:

$$P(M, S, I, R, B, F) = P(M|S, I, R, B, F)P(S|I, R, B, F)P(I|R, B, F)P(R|B, F)P(B|F)P(F) \ .$$

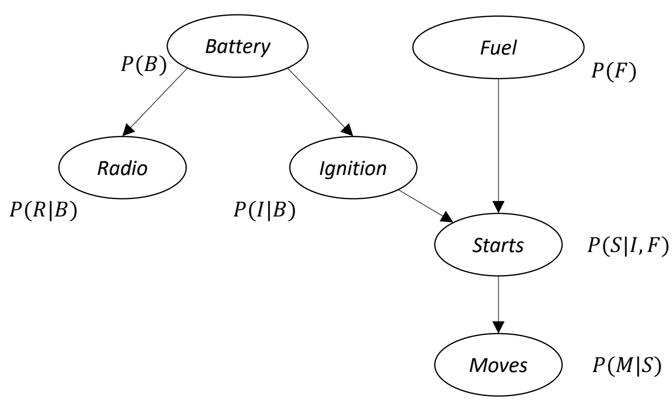
   (b) To identify suitable conditional independence assumptions, consider the conditional distributions in the above expression of the chain rule, from right to left:

   - $P(B|F)$: the state of the battery and of the fuel tank can be considered independent on each other: $P(B|F) = P(B)$ (this is an *absolute* independence relation, not a conditional one);
   - $P(R|B, F)$: whether the radio works or not, given the state of the battery, is independent on the state of the fuel tank: $P(R|B, F) = P(R|B)$;
   - $P(I|R, B, F)$: the working of the ignition system, given the state of the battery, is independent on the state of the radio and of the fuel tank: $P(I|R, B, F) = P(I|B)$;
   - $P(S|I, R, B, F)$: given the state of the fuel tank and of the ignition system, the working of the engine is independent on the state of the radio and of the battery (note that the battery affects the engine only *indirectly*, through the ignition system): $P(S|I, R, B, F) = P(S|I, F)$;
   - $P(M|S, I, R, B, F)$: given the state of the engine, whether the car moves or not is independent on all the other factors: $P(M|S, I, R, B, F) = P(M|S)$.

   Accordingly, the expression of the joint distribution becomes:

$$P(M, S, I, R, B, F) = P(M|S)P(S|I, F)P(I|B)P(R|B)P(B)P(F) \ .$$

   (c) The corresponding Bayesian network is shown below.



   (d) Remember that, to specify the *unconditional* distribution $P(X)$ of any Boolean random variable X, only one value needs to be estimated: either $P(X = true)$ or $P(X = false)$, since the other value is determined by the constraint $P(X = true) + P(X = false) = 1$. To define the *conditional* distribution $P(X|Y_1, \ldots, Y_n)$, given the values of n Boolean random variables $Y_1, \ldots, Y_n$, either the probability that $X = true$ or the probability that $X = false$ need to be estimated, for *each* of the 2^n possible combinations of values of $Y_1, \ldots, Y_n$.

Accordingly, taking into account that all the considered random variables are Boolean:

- $P(F)$ and $P(B)$ require the estimation of one probability value each, therefore 2 values are required in total;
- $P(R|I)$ requires one value for $I = true$ and one for $I = false$; similarly for $P(R|B)$ and $P(M|S)$; therefore, these distributions require 6 values in total;
- $P(S|I, F)$ requires one value for each of the four combinations of values of I and F, therefore 4 values in total.

The joint distribution of the considered 6 Boolean variables can therefore be specified through $2 + 6 + 4 = 12$ probability values, thanks to the above conditional independence assumptions, instead of $2^6 - 1 = 63$ values.

Some values of the above probability distributions can be set *a priori*, based on causal knowledge on the corresponding events. In particular, the ignition system and the radio cannot work, if the battery is dead, i.e., $P(I = true|B = false) = P(R = true|B = false) = 0$. Similarly, if the ignition system does not work or the fuel tank is empty, the engine cannot fire: $P(S = true|I = false, F) = 0$, and $P(S = true|I, F = false) = 0$.

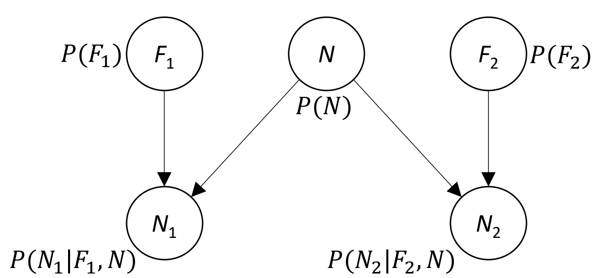
On the other hand, the ignition system may not work even when the battery is *not* dead, due to several possible causes that one may not know or may not be willing to consider *explicitly*, such as a broken fuel pump or a fault in the ignition system itself. Therefore, $P(I = true|B = true)$ should *not* be set to 1, to account for *any* other possible cause that prevent the ignition system from working. Analogously, the engine may not fire even when the ignition system *does* work and the fuel tank is *not* empty; therefore, the corresponding probability, $P(S = true|I = true, F = true)$, should be lower than 1.

2. (a) Five random variables can be used to describe the relevant information:

   - M_1 and M_2: the number of stars counted by the two astronomers: their domain is the set of natural numbers $\{0, 1, 2, \ldots\}$;
   - F_1 and F_2: Boolean random variables representing whether the two telescopes are out of focus ($true$) or not ($false$);
   - N, the actual (unknown) number of stars in the region of the sky under observation: its domain is the set of natural numbers.

   (b) The actual number of stars N and the states of the two telescopes (F_1 and F_2) can be considered as root causes; it can also be assumed that they do not affect each other. The number of stars N_1 estimated by the first astronomer is directly influenced only by the actual number of stars N and by the state of his or her telescope, F_1, but not by the number of stars estimated by the other astronomer, N_2 (assuming they do not communicate with each other); similarly, N_2 is directly influenced by only N and F_2.

   (c) Considering the random variables ordered as F_1, N, F_2, N_1, N_2 (in agreement with the above causal relations), the corresponding Bayesian network is:



   The encoded conditional independence assumptions are:

   - N and F_1 are independent on each other: $P(N|F_1) = P(N)$;
   - F_2 is independent on both N and F_1: $P(F_2|N, F_1) = P(F_2)$;

- N_1 is conditionally independent on F_2, given N and F_1: $P(N_1|F_2, N, F_1) = P(N_1|N, F_1)$;
- N_2 is conditionally independent on N_1 and F_1, given N and F_2: $P(N_2|N_1, F_2, N, F_1) = P(N_2|F_2, N)$.

(d) The corresponding expression of the joint distribution is:

$$P(N_2, N_1, F_2, N, F_1) = P(N_2|N, F_2)P(N_1|N, F_1)P(F_2)P(N)P(F_1) \; .$$

The conditional distributions $P(N_1|N, F_1 = false)$ and $P(N_2|N, F_2 = false)$ (i.e., the distributions of the number of stars estimated by either astronomer when the corresponding telescope is not out of focus, whatever the actual number of stars is) should be greater than zero when the estimate is *wrong*, i.e, when $N_1 \neq N$ or $N_2 \neq N$. As an example:

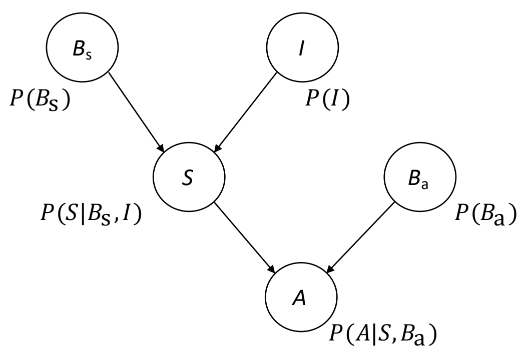
$$P(N_1 = 1000|N = 1050, F_1 = false) > 0 \; .$$

The reason is that the estimated number of stars may be wrong due to other possible causes beside the telescope being out of focus, that are not explicitly taken into account in this problem formulation (e.g., the sky may be not perfectly clear when the observation is made), or may be even unknown.

3. (a) This domain can be described by five Boolean random variables denoting the presence of someone inside the car (I), the state (dead or not dead) of the batteries powering the sensor (B_s) and the alarm (B_a), the state (detection or no detection) of the sensor (S) and the state (sounding or not sounding) of the alarm (A).

(b) It is reasonable to assume that the "root causes" correspond to the presence of someone inside the car (I) and to the states of the two batteries (B_s and B_a), and that these events do not directly affect each other. The state of the sensor (S) is directly influenced only by the state of its battery (B_s) and by the presence of someone inside the car (I). The state of the alarm (A) is directly influenced only by the states of its battery (B_a) and of the sensor (S).

(c) Considering the random variables ordered as B_s, I, S, B_a, A (in agreement with the above causal relations), the corresponding Bayesian network is:



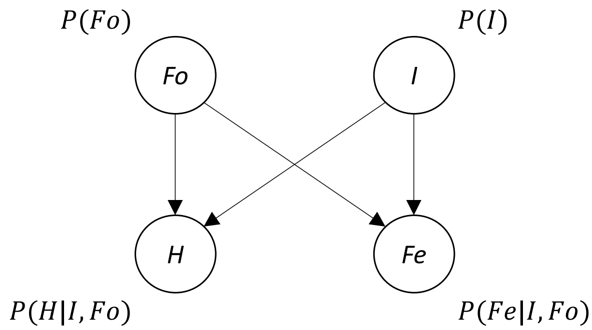
The encoded conditional independence assumptions are:

- I and B_s are independent on each other: $P(I|B_s) = P(I)$;
- B_a is independent on S, I and B_s: $P(B_a|S, I, B_a) = P(B_a)$;
- A is conditionally independent on I and B_s, given B_a and S: $P(A|B_a, S, I, B_s) = P(A|B_a, S)$.

(d) The corresponding expression of the joint distribution is:

$$P(A, B_a, S, I, B_s) = P(A|B_a, S)P(B_a)P(S|I, B_s)P(I)P(B_s) \; .$$

4. (a) Four Boolean random variables can be used to represent the occurrence of the two symptoms (H for headache and Fe for fever) and of the two health problems (I for influenza and Fo for food poisoning). Influenza and food poisoning can be considered as the root causes. Each of the symptoms is directly affected by both health problems. On the other hand, headache and fever can be assumed not to directly affect each other. Analogously, food poisoning and influenza do not directly affect each other. A possible order between the random variables is therefore: Fo, I, H, Fe. The corresponding Bayesian network is shown below.



The encoded conditional independence assumptions are:

- I is independent on Fo: $P(I|Fo) = P(I)$;
- Fe is conditionally independent on H, given I and Fo: $P(Fe|H, I, Fo) = P(Fe|I, Fo)$.

(b) The joint distribution function corresponding to the above Bayesian network is:

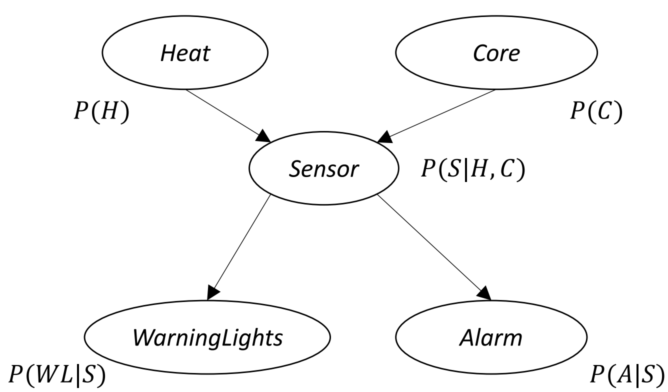
$$P(Fe, H, I, Fo) = P(Fe|I, Fo)P(H|I, Fo)P(I)P(Fo) \ .$$

(c) The probability to compute is $P(I = t|H = t)$. Using the standard exact inference procedure one obtains:

$$\begin{aligned} P(I = t|H = t) &= \frac{P(I = t, H = t)}{P(H = t)} \\ &= \frac{\sum_{fe,fo} P(Fe = fe, H = t, I = t, Fo = fo)}{\sum_{fe,i,fo} P(Fe = fe, H = t, I = i, Fo = fo)} \\ &= \frac{\sum_{fe,fo} P(Fe = fe|I = t, Fo = fo)P(H = t|I = t, Fo = fo)P(I = t)P(Fo = fo)}{\sum_{fe,i,fo} P(Fe = fe|I = i, Fo = fo)P(H = t|I = i, Fo = fo)P(I = i)P(Fo = fo)} \ . \end{aligned}$$

5. (a) This domain can be represented using five Boolean random variables: $Core$ (the core temperature exceeding the safety threshold), $Sensor$ (the sensor detecting a core overheating), $Heat$ (the external temperature being higher than some specific value), $WarningLights$ (warning lights flashing) and $Alarm$ (alarm sounding).

The "root" causes are the core temperature and the external temperature. They can be assumed not to directly infuence each other. On the other hand, they both directly influence only the sensor measurement. In turn, the sensor measurement directly affects both the warning lights and the alarm, whereas the latter two can be assumed not to directly influence each other, since they are implemented as physically separated systems. Accordingly, a possible order between the random variables is: $Heat, Core, Sensor, WarningLights, Alarm$. The corresponding Bayesian network is shown below.



Denoting the random variables with their initial letters, the encoded conditional independence assumptions are:

- C is independent on H: $P(C|H) = P(C)$;
- WL is independent on both C and H, given S: $P(WL|S, C, H) = P(WL|S)$;
- A is independent on WL, C and H, given S: $P(A|WL, S, C, H) = P(A|S)$.

(b) The expression of the joint distribution function corresponding to the above Bayesian network is:

$$P(A, WL, S, C, H) = P(A|S)P(WL|S)P(S|C, H)P(C)P(H) \ .$$

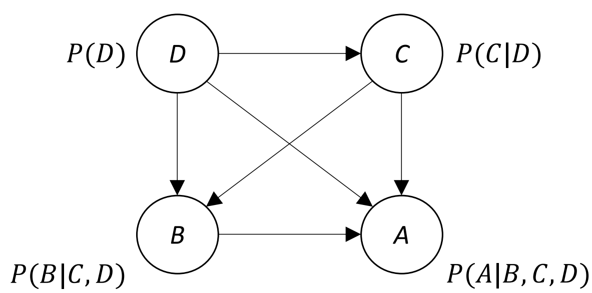
(c) The probability to compute is $P(C = true|WL = true)$. Using the standard exact inference procedure one obtains (the value *true* is shortened to t in the expressions below):

$$\begin{aligned} P(C = t|WL = t) &= \frac{P(C = t, WL = t)}{P(WL = t)} \\ &= \frac{\sum_{a,s,h} P(A = a, WL = t, S = s, C = t, H = h)}{\sum_{a,s,c,h} P(A = a, WL = t, S = s, C = c, H = h)} \\ &= \frac{\sum_{a,s,h} P(A = a|S = s)P(WL = t|S = s)P(S = s|C = t, H = h)P(C = t)P(H = h)}{\sum_{a,s,c,h} P(A = a|S = s)P(WL = t|S = s)P(S = s|C = c, H = h)P(C = c)P(H = h)} \ . \end{aligned}$$

6. By applying the chain rule in the considered order, the joint distribution can be rewritten as:

$$P(A, B, C, D) = P(A|B, C, D)P(B|C, D)P(C|D)P(D) \ .$$

The corresponding Bayesian network is:



Note that the above graph is *fully connected*, i.e., there is an (oriented) arc between *every* pair of nodes. This is a general characteristic of any Bayesian network when no conditional independence assumption can be made on the corresponding conditional distributions.

The number of probability values that have to be estimated to specify the distribution functions associated to the nodes of the above Bayesian network is:

- 1 value for $P(D)$,
- 2 values for $P(C|D)$,
- 4 values for $P(B|C, D)$,
- 8 values for $P(A|B, C, D)$,

for a total of 15 values. Note that this is the same number of probability values that have to be estimated for the full joint distribution of the 4 random variables, $P(A, B, C, D)$, that is, $2^4 - 1 = 15$. In general, for n Boolean random variables the number of probability values to be estimated is $\sum_{k=1}^{n} 2^{k-1} = 2^n - 1$. In general, the number of required probability values for discrete random variables increases *exponentially* in the number of variables. Conditional independence assumptions are useful in practical applications to reduce the effort required to estimate the joint distribution expressed using the chain rule.

7. (a) The expression of the joint distribution function corresponding to the Bayesian network is:

$$P(D, C, B, A) = P(D|A)P(C|B, A)P(B)P(A) \ . \tag{1}$$

(b) Taking into account the specified order between the random variables, the independence assumptions encoded by the Bayesian network are:

- B is independent on A: $P(B|A) = P(B)$;
- D is conditionally independent on C and B, given A: $P(D|C, B, A) = P(D|A)$.

(c) The expression of the considered probability can be derived using the standard exact inference procedure as follows (the value *true* is shortened to t in the expressions on the right):

$$\begin{aligned} P(A = true|C = true, D = true) &= \frac{P(A = t, C = t, D = t)}{P(C = t, D = t)} \\ &= \frac{\sum_b P(A = t, B = b, C = t, D = t)}{\sum_{a,b} P(A = a, B = b, C = t, D = t)} \\ &= \frac{\sum_b P(D = t|A = t)P(C = t|B = b, A = t)P(B = b)P(A = t)}{\sum_{a,b} P(D = t|A = a)P(C = t|B = b, A = a)P(B = b)P(A = a)} \ . \end{aligned}$$

(d) The rejection sampling algorithm works by first generating a given number (say, N) of samples of the four random variables, according to their joint distribution as defined by the Bayesian network, following the topological order A, B, C, D. This is an example of how a *single* sample is generated:

- a sample from $P(A)$ is drawn: assume $A = false$ is obtained;
- a sample from $P(B)$ is drawn: assume $B = false$ is obtained;
- a sample from $P(C|B = false, A = false)$ is drawn: assume $C = true$ is obtained;
- a sample from $P(D|A = false)$ is drawn: assume $D = false$ is obtained.

The corresponding sample is $A = false, B = false, C = true, D = false$. Note that it *does not* agre with the evidence $C = true, D = true$.

Next, all the samples that do not agree with the evidence are rejected. Let $N_{\mathrm{E}} \leq N$ denote the number of the remaining samples; among them, let $N^* \leq N_{\mathrm{E}}$ denote the number of samples that correspond to the event of interest, $A = true$. The probability $P(A = true|C = true, D = true)$ is then estimated as N^*/N_{E}.